International Journal of Theoretical Physics, Vol. 33, No. 11, 1994

Cosmic Spinor Fields and the Early Universe

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Received January 17, 1994

The energy production through expansion of the universe is studied for the Dirac spinor field in all three types of Robertson–Walker universes. Only in the open case is the matter production unlimited (closed universe: limited; flat universe: impossible). The physical properties of the cosmological solutions to the Dirac equation over any RW background are studied in detail.

1. INTRODUCTION

In modern cosmology, the common belief is that our universe in its evolution was subject to two essentially different phases, namely the primeval phase and the subsequent standard phase in which we live today. Both phases are assumed to be separated by a certain phase transition, during which the exotic physics of the primeval era was changed into the usual form of physics as known today. As a natural consequence of this picture of the universe's history, the cosmic dynamics of the present standard phase is satisfactorily described by well-established theories (Kolb and Turner, 1990), such as relativistic thermodynamics or Einstein's theory of gravity based upon his famous field equation

$$R_{\mu\nu} - \frac{1}{2} R G_{\mu\nu} = 8\pi \frac{{L_p}^2}{\hbar c} T_{\mu\nu} \tag{1.1}$$

However, it appears almost trivial to point out that this standard cosmology runs into serious difficulties when trying to clarify its own initial conditions, e.g., the questions (i) where does all the huge energy content of the present-day universe come from? (ii) what was the origin of the initial outward push for the expansion lasting up to the present day? (iii) why is the cosmological principle obeyed so accurately (on large scales)? The fact

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0020-7748/94/1100-2157$07.00/0

that standard cosmology is completely unable to give a satisfying answer to such questions is readily realized by writing down the Einstein equations (1.1) for the typical cosmological situation of a Robertson–Walker universe where the energy-momentum density $T_{\mu\nu}$ of matter must have its "cosmological shape":

$$T_{\mu\nu} = \mathscr{M} b_\mu b_\nu - \mathscr{P} \mathscr{B}_{\mu\nu} \tag{1.2}$$
$$(\mathscr{B}^\mu{}_\nu b_\mu = 0, \qquad G_{\mu\nu} = \mathscr{B}_{\mu\nu} + b_\mu b_\nu)$$

Here, the energy density $\mathscr{M}$ and pressure $\mathscr{P}$ are assumed to be homogeneous (*cosmological principle*) and therefore depend exclusively upon the cosmic time θ according to Einstein's equation (1.1):

$$\frac{d}{d\theta}(\mathscr{M}\mathscr{R}^3) = -\mathscr{P}\frac{d\mathscr{R}^3}{d\theta} \tag{1.3a}$$

$$\frac{1}{\mathscr{R}}\frac{d^2\mathscr{R}}{d\theta^2} = -4\pi\frac{L_p{}^2}{\hbar c}\left(\mathscr{P} + \frac{1}{3}\mathscr{M}\right) \tag{1.3b}$$

($\mathscr{R}$ is the radius of the universe).

Thus, during the standard phase where the energy density $\mathscr{M}$, pressure $\mathscr{P}$, and expansion rate $H = \dot{\mathscr{R}}/\mathscr{R}$ are assumed to be always nonnegative, equations (1.3) say that both the energy $\mathscr{M}\mathscr{R}^3$ in a comoving 3-cell of size $\mathscr{R}^3$ and the expansion rate H are always *decreasing*! Consequently, the standard cosmology is completely unable to explain the origin of the primordially high energy per 3-cell and how the expansion rate H achieved its primevally high value!

In order to remedy these and other deficiencies of the standard cosmological model (e.g., horizon and flatness problems, etc.), it is very suggestive to resort to the hypothesis that the standard phase was preceded by some quantum era during which the matter energy was present in the form of a coherent quantum state ψ extending over the whole universe (being still incredibly tiny at that epoch). Clearly, such a hypothesis will gain credibility only through successfully solving (at least some of) the puzzles of the standard cosmology, as mentioned above. That this is in principle feasible has been demonstrated through the new paradigm of inflation (Guth, 1981; Blau and Guth, 1987; Abbott and Pi, 1986), where the global quantum state ψ refers to a real scalar field which, during the inflationary phase, is in its "false" vacuum state and obeys the equation of state

$$\mathscr{P} = -\mathscr{M} = \text{const} \tag{1.4}$$

Evidently, such an exotic equation of state can account for *both* the increase of energy in a 3-cell according to (1.3a) *and* the exponential growth of the universe's size $\mathscr{R}$ according to (1.3b). In this way, the inflation model can successfully overcome several of the weak points of the standard cosmological model, and it is merely the precise nature of the phase transition from the inflationary into the standard phase which has caused some controversy and still must await its final clarification (Hawking, 1990; Penrose, 1989).

Now we have arrived at the point of the present paper; if it is some particle field in its global quantum state which is responsible for all the exotic physics occurring in the prestandard epoch, we have to ask which of the known particle fields is the right candidate. Historically, the *scalar* field was preferred (Guth, 1981), because it emerged in the grand unified theories being studied intensively in connection with cosmological problems. However, it seems meaningful to ask whether other particle fields can do the same job (or even better?). An alternative to the scalar field has recently been proposed in the form of the Dirac spinor field (Sorg, 1992*a*), because this is the most dominant particle field in nature and it would appear somewhat strange for this dominance not to have been present also during the prestandard phase! Indeed, it can be demonstrated (Sorg, 1992*b*,*c* 1993; Mattes and Sorg, 1993; Ochs and Sorg, n.d.; Mattes *et al.*, 1993) that the spinor field is able to create energy *ex nihilo* in a nonsingular way (for vanishing radius $\mathscr{R} \to 0$) if it is interacting with gravity via the Einstein equation (1.1) ($\leadsto$ minimal coupling). However, there are significant differences from the inflation model:

(i) Whereas the mechanism of inflation via a scalar field works in any type of universe (closed, flat, open), the energy production by a spinor field is effectively possible exclusively in an *open* universe (limited energy production in a *closed* universe and no production in a *flat* universe).

(ii) Whereas in the inflation model, based upon the scalar field, energy is created during the *exponential growth* of radius $\mathscr{R}$, the Dirac spinor field creates energy through *oscillations* in size $\mathscr{R}$.

Such a "*cosmic pumping*" process may be preliminarily explained as follows (Mattes *et al.*, 1993): The pressure $\mathscr{P}$ of the Dirac field ψ looks like $\mathscr{P} \sim (\cos \chi)/\mathscr{R}$ and thus is negative only when the relative phase shift χ between the positive and negative energy components of ψ are in the range $\pi/2 < \chi < 3\pi/2$. Consequently, when this occurs, we must have a small radius $\mathscr{R}$ in order to get an effective energy production. But simultaneously the negative pressure $\mathscr{P}$ blows up the universe according to (1.3b) and thus spoils the effectiveness of energy production. As a consequence, the universe has to pass several phases of minimal extension (bounces) in order to increase its energy content with every bounce up to the large value being presently observed.

In the present paper a better understanding of this "cosmic pumping" process is sought by studying in detail the physical properties of the cosmological solutions to the Dirac equation in all three types of Robertson–Walker universes. We will examine the (pseudo) scalar density, the (axial) current, and the polarization. The Robertson–Walker geometry is arbitrarily prescribed and not determined from the Einstein equation (1.1), because this is possible exclusively in the *open* universe (Mattes and Sorg, 1993; Mattes *et al.*, 1993). Similarly as in the inflationary scenario, the precise nature of the phase transition into the standard phase must be left unclear in the present state of the theory. Since such a phase transition is expected to consist in the decay of the global quantum state into the decoherent wave functions of the individual particles (of the big bang plasma), it seems to us that one first has to solve the general problem of decoherence in quantum theory, before one can attack the problem of the cosmic phase transition!

Our procedure is the following:

1. First we obtain the solutions ψ to the Dirac equation of the "cosmological type," i.e., the corresponding energy-momentum density $T_{\mu\nu}$ must be of a form similar to (1.2) with homogeneous $\mathcal{M}$ and $\mathcal{P}$ (Section 2).
2. Next we establish the dynamical equations for the physical densities of the spinor field by recasting the Dirac equation into a relativistic Schrödinger equation (Section 3).
3. Then we try to solve the puzzles of the standard cosmology, mentioned below (1.1), by discussing the specific properties of the spinor densities. We study matter production and the question of the cosmological principle in all three types of universe (Sections 4–6).
4. We collect the results in a survey table (Section 7).

2. EXACT COSMOLOGICAL SOLUTIONS

Assuming that matter energy during the prestandard phase was present in some form of coherent quantum state $\psi(x)$ extending over the whole universe and in agreement with the cosmological principle, we first have to look for the corresponding solutions of the Dirac equation

$$i\hbar \cdot \gamma^{\mu}\mathcal{D}_{\mu}\psi = Mc\psi \qquad (\mathcal{D}_{\mu} = \partial_{\mu} + \mathcal{A}_{\mu}) \tag{2.1}$$

in anyone of the three types of FRW universe (closed, flat, open). This problem has already been solved (Sorg, 1992*a*,*b*, 1993; Mattes and Sorg, 1993) and we merely collect the results as our point of departure for the subsequent investigations.

First, one recasts the Dirac equation (2.1) into the form of a "relativistic Schrödinger equation" (Sorg, 1992*a*):

$$i\hbar c\mathcal{D}_\mu\psi = \mathcal{H}_\mu\psi \tag{2.2}$$

and then one tries for the Hamiltonian $\mathcal{H}_\mu$ the "cosmological ansatz" ${}^{(s)}\mathcal{H}_\mu$ (being compatible with the exact Robertson–Walker symmetry):

$$\begin{aligned}(\hbar c)^{-1}\cdot{}^{(s)}\mathcal{H}_\mu &= \frac{1}{4}m\gamma_\mu + \frac{3}{2}ib_\mu\{N\cdot 1 - \tilde{N}\cdot\varepsilon\}\\ &\quad + (4b_\mu b_\lambda - G_{\mu\lambda})\{W\cdot 1 + \tilde{W}\cdot\varepsilon\}\cdot\gamma^\lambda\\ &\quad - ib^\lambda\{N\cdot\Sigma_{\mu\nu} + \tilde{N}\cdot{}^*\Sigma_{\mu\lambda}\}\end{aligned} \tag{2.3}$$

($m := Mc/\hbar$). Observe that entering this ansatz are nothing else than the Clifford algebra elements $\{\gamma_\mu, \varepsilon, \Sigma_{\mu\nu}, {}^*\Sigma_{\mu\nu} = -\varepsilon\cdot\Sigma_{\mu\nu}\}$ as kinds of absolute (i.e., nondynamical) objects, as well as the Hubble flow b_μ ($b^\mu b_\mu = +1$) and finally the homogeneous scalar fields

$$\begin{aligned}N &= {}^{(c)}N - i\,{}^{(r)}N\\ \tilde{N} &= {}^{(c)}\tilde{N} - i\,{}^{(r)}\tilde{N}\\ W &= {}^{(r)}W + i\,{}^{(c)}W\\ \tilde{W} &= {}^{(r)}\tilde{W} + i\,{}^{(c)}\tilde{W}\end{aligned} \tag{2.4}$$

Clearly, the Hubble flow b_μ is the only possible vector field admitted by the Robertson–Walker symmetry. Denoting its orthogonal projector by $\mathcal{B}_{\mu\nu}$ ($:= G_{\mu\nu} - b_\mu b_\nu$), we have that the cosmological form of any energy-momentum density $T_{\mu\nu}$ which is consistent with the RW symmetry reads

$$T_{\mu\nu} = \mathcal{M}b_\mu b_\nu - \mathcal{P}\mathcal{B}_{\mu\nu} \tag{2.5}$$

where the energy density $\mathcal{M}$ and the pressure $\mathcal{P}$ must be homogeneous,

$$\mathcal{B}^\mu{}_\nu\,\partial_\mu\mathcal{M} = 0 \tag{2.6a}$$

$$\mathcal{B}^\mu{}_\nu\,\partial_\mu\mathcal{P} = 0 \tag{2.6b}$$

The conditions (2.5) and (2.6) will in general not be satisfied if we build up the energy-momentum density ${}^{(D)}T_{\mu\nu}$ for the Dirac spinor field

$${}^{(D)}T_{\mu\nu} = \frac{1}{2}\bar{\psi}(\gamma_\mu\cdot\mathcal{H}_\nu + \bar{\mathcal{H}}_\nu\cdot\gamma_\mu)\psi \tag{2.7}$$

by means of the cosmological Hamiltonian ${}^{(s)}\mathcal{H}_\mu$, (2.3), and by the corresponding solution $\psi(x)$ of the Dirac equation (2.1). Although the cosmological principle as expressed by equations (2.5), (2.6) can be strictly satisfied by imposing certain additional restrictions upon ${}^{(s)}\mathcal{H}_\mu$ [cf. (2.3)]

(Mattes and Sorg, 1993)—as would be necessary for solving the *coupled* Dirac–Einstein system (1.1) + (2.1)—we need here merely the weaker condition

$$T_{\mu\nu} = \mathcal{M} b_\mu b_\nu - \mathcal{P}\mathcal{B}_{\mu\nu} + V_\nu b_\mu \tag{2.8a}$$

$$V_\mu b^\mu = 0 \tag{2.8b}$$

The spacelike vector field V_μ has to guarantee the validity of the work-energy theorem (1.3a). The reason is that we are mainly interested in the energy production on a given RW background; but we want this production to be homogeneous and isotropic in agreement with the cosmological principle [cf. (2.6)].

After having specified the Hamiltonian $\mathcal{H}_\mu$ in a first step, we have now to ensure the existence of the corresponding solution $\psi(x)$ of the Dirac equation (2.1) in a second step. As shown previously (Sorg, 1992*a*), the solutions of the relativistic Schrödinger equation (2.2) fit also the Dirac equation (2.1) whenever the Hamilton $\mathcal{H}_\mu$ obeys the following two conditions:

$$\mathcal{D}_\mu \mathcal{H}_\nu - \mathcal{D}_\nu \mathcal{H}\mu + \frac{i}{\hbar c}[\mathcal{H}_\mu, \mathcal{H}_\nu] = i\hbar c \mathcal{F}_{\mu\nu} \tag{2.9}$$

$$\mathcal{D}^\mu \mathcal{H}_\mu - \frac{i}{\hbar c}\{(\mathcal{H}_\mu \cdot \mathcal{H}^\mu) - (Mc^2)^2 \cdot 1\} = -i\hbar c \Sigma^{\mu\nu} \mathcal{F}_{\mu\nu} \tag{2.10}$$

Here $\mathcal{F}_{\mu\nu}$ is the curvature of the connection $\mathcal{A}_\mu$ entering the covariant derivative $\mathcal{D}_\mu$, (2.1); i.e., $\mathcal{F}_{\mu\nu}$ is the space-time curvature of our Robertson–Walker universe. Now, introducing the cosmological Hamiltonian ${}^{(s)}\mathcal{H}_\mu$, (2.3), into these conditions (2.9), (2.10) yields a system of constraints for those scalar fields (2.4). These constraints are partly of dynamical and partly of kinematical character. The dynamical part is (a dot denotes differentiation with respect to cosmic time θ)

$$\dot{W} + 4HW = -3N\left(W + \frac{m}{12}\right) \tag{2.11a}$$

$$\dot{\tilde{W}} + 4H\tilde{W} = -3N\tilde{W} \tag{2.11b}$$

$$\dot{\tilde{N}} + H\tilde{N} = 0 \tag{2.11c}$$

$$(\dot{N} + \dot{H}) - N(N + H) = 16(W^2 + \tilde{W}^2) - 4mW - \tilde{N}^2 - \frac{\sigma}{\mathcal{R}^2} \tag{2.11d}$$

and the kinematical part is

$$\tilde{N} \cdot \tilde{W} = 0 \tag{2.12a}$$

$$\tilde{N}\left(W - \frac{m}{4}\right) = 0 \tag{2.12b}$$

$$\tilde{N}(N+H)=0 \tag{2.12c}$$

$$(N+H)^2=\frac{\sigma}{\mathscr{R}^2}-4\left(W-\frac{m}{4}\right)^2-4\tilde{W}^2+\tilde{N}^2 \tag{2.12d}$$

The interesting point here is that entering these constraints are, besides the Hubble expansion rate H, also the topological index σ, which specifies the nature of the space-time foliation: $\sigma=+1$: open universe; $\sigma=0$: flat universe; and $\sigma=-1$: closed universe. As a consequence, the existence and type of solutions to the Dirac equation will strongly depend upon this foliation index and we have to expect quite different physical properties of the matter field ψ according to which kind of topology for the universe we are considering!

The integrability conditions (2.11), (2.12) are necessary but not sufficient; but, as can be shown easily, the missing supplement for sufficiency is

$$[\mathscr{L}_\mu, \mathscr{L}_\nu]=0 \tag{2.13}$$

where $\mathscr{L}_\mu$ is the anti-Hermitian part of the Hamiltonian:

$$\begin{aligned}\mathscr{H}_\mu&=\hbar c(\mathscr{K}_\mu+i\mathscr{L}_\mu)\\ \mathscr{K}_\mu&=\bar{\mathscr{K}}_\mu=\frac{1}{2\hbar c}(\mathscr{H}_\mu+\bar{\mathscr{H}}_\mu)\\ \mathscr{L}_\mu&=\bar{\mathscr{L}}_\mu=\frac{1}{2i\hbar c}(\mathscr{H}_\mu-\bar{\mathscr{H}}_\mu)\end{aligned} \tag{2.14}$$

Thus, introducing our cosmological ansatz (2.3) into the supplementary condition (2.13) readily yields the additional constraints

$${}^{(c)}\tilde{N}\cdot{}^{(r)}\tilde{W}-{}^{(r)}N\cdot{}^{(c)}W=0 \tag{2.15a}$$

$${}^{(c)}W\cdot{}^{(r)}\tilde{N}=0 \tag{2.15b}$$

$${}^{(c)}\tilde{N}\cdot{}^{(c)}W-{}^{(r)}N\cdot{}^{(r)}\tilde{W}=0 \tag{2.15c}$$

$${}^{(r)}\tilde{W}\cdot{}^{(r)}\tilde{N}=0 \tag{2.15d}$$

$${}^{(r)}N\cdot{}^{(r)}\tilde{N}=0 \tag{2.15e}$$

$$({}^{(r)}\tilde{N})^2=({}^{(r)}N)^2 \tag{2.15f}$$

$$({}^{(r)}\tilde{W})^2=({}^{(c)}W)^2 \tag{2.15g}$$

This is the desired result: any *cosmological* solution for the Dirac equation (2.1) on a given RW background is determined by equations (2.11), (2.12), and (2.15). Clearly, entering this are the characteristic features of such a RW geometry, namely the topological index σ, the expansion rate $H=\dot{\mathscr{R}}/\mathscr{R}$, and the radius of the universe $\mathscr{R}$. For the subsequent discussion, let us begin with the simplest case: $\tilde{N} \neq 0$. As is readily deduced from the kinematical constraints (2.12), the only possibility for $\tilde{N}$ is here

$$(\tilde{N})^2 \equiv ({}^{(c)}\tilde{N})^2 \Rightarrow -\frac{\sigma}{\mathscr{R}^2} \qquad ({}^{(r)}\tilde{N} \Rightarrow 0) \tag{2.16}$$

i.e., this solution can exist only in a *closed* universe ($\sigma = -1$). Further, as is readily checked, the dynamical equations (2.11) and the supplementary conditions (2.15) are also satisfied by (2.16) with the remaining scalars being found as

$$\tilde{W} \Rightarrow 0 \tag{2.17a}$$

$${}^{(r)}W \Rightarrow \frac{m}{4}, \qquad {}^{(c)}W \Rightarrow 0 \tag{2.17b}$$

$${}^{(c)}N \Rightarrow -H, \qquad {}^{(r)}N \Rightarrow 0 \tag{2.17c}$$

Thus, our first solution is given by the Hamiltonian

$$\mathscr{H}_\mu \Rightarrow {}^{(\sim)}\mathscr{H}_\mu = \hbar c\Big\{mb_\mu(b^\lambda\gamma_\lambda) - \frac{3}{2}\,ib_\mu\Big(H\cdot 1 \pm \frac{1}{\mathscr{R}}\cdot\varepsilon\Big) + ib^\lambda\Big(H\cdot\Sigma_{\mu\lambda} \mp \frac{1}{\mathscr{R}}\cdot{}^*\Sigma_{\mu\lambda}\Big)\Big\} \tag{2.18}$$

and is possible exclusively in a closed universe ($\sigma = -1$).

For obtaining the *remaining solutions*, we return to the kinematical conditions (2.12) and take now the alternative case: $\tilde{N} = 0$. Here equations (2.12) are automatically satisfied, whereas the supplementary conditions (2.15) merely yield

$${}^{(r)}N \Rightarrow 0 \tag{2.19}$$

But this implies that the right-hand sides of both equations (2.11d) and (2.12d) must be real, i.e.,

$${}^{(c)}W \Rightarrow 0 \tag{2.20a}$$

$${}^{(r)}\tilde{W}\cdot{}^{(c)}\tilde{W} \Rightarrow 0 \tag{2.20b}$$

and now the last supplementary condition (2.15) says that

$${}^{(r)}\tilde{W} \Rightarrow 0 \tag{2.21}$$

With these results, the last kinematical equation (2.12d) reduces to

$$({}^{(c)}N + H)^2 + 4\Big({}^{(r)}W - \frac{m}{4}\Big)^2 - 4({}^{(c)}\tilde{W})^2 = \frac{\sigma}{\mathscr{R}^2} \tag{2.22}$$

and does indeed admit all three topological cases $\sigma = 0, \pm 1$. However, it is clear that the topological index cannot change during the expansion of the universe and we can resort to the following parametrizations:

$$\left.\begin{aligned} {}^{(c)}N + H &= \frac{\cosh \xi \cdot \sin \Phi}{\mathscr{R}} \quad &(2.23a)\\ 2\left({}^{(r)}W - \frac{m}{4}\right) &= \frac{\cosh \xi \cdot \cos \Phi}{\mathscr{R}} \quad &(2.23b)\\ 2\,{}^{(c)}\tilde{W} &= \frac{\sinh \xi}{\mathscr{R}} \quad &(2.23c)\end{aligned}\right\}\sigma = +1$$

for $\sigma = +1$ or

$$\left.\begin{aligned} {}^{(c)}N + H &= \frac{\sinh \xi \cdot \sin \Phi}{\mathscr{R}} \quad &(2.24a)\\ 2\left({}^{(r)}W - \frac{m}{4}\right) &= \frac{\sinh \xi \cdot \cos \Phi}{\mathscr{R}} \quad &(2.24b)\\ 2\,{}^{(c)}\tilde{W} &= \pm\frac{\cosh \xi}{\mathscr{R}} \quad &(2.24c)\end{aligned}\right\}\sigma = -1$$

for $\sigma = -1$. Observe here that the flat case ($\sigma = 0$) is contained in the open case (2.23) in the limit of an infinite universe ($\mathscr{R} \to \infty$) when the "timelike" hyperboloid shrinks to the "light cone." This light cone itself separates the "spacelike" hyperboloid ($\sigma = -1$) from the timelike one ($\sigma = +1$) [furthermore, the previously discussed situation $\xi \equiv 0$ (Sorg, 1992*a*,*b*, 1993; Mattes and Sorg, 1993) is also a particular subcase of (2.23)]. The light-cone parametrization is ($\sigma = 0$)

$$\left.\begin{aligned} {}^{(c)}N + H &= 2\,{}^{(c)}\tilde{W} \cdot \sin \Phi \quad &(2.25a)\\ {}^{(r)}W - \frac{m}{4} &= {}^{(c)}\tilde{W} \cdot \cos \Phi \quad &(2.25b)\end{aligned}\right\}\sigma = 0$$

Of course, once the parameters ξ and Φ are introduced one wants to deduce their dynamical equations from the Hamiltonian dynamics (2.11). This can be easily achieved by simply inserting the parametrizations (2.23)–(2.25) into the original dynamics (2.11). The results are, for the open case $\sigma = +1$, (2.23),

$$\left.\begin{aligned} \dot{\xi} &= -3\,\frac{\sinh \xi \cdot \sin \Phi}{\mathscr{R}} \quad &(2.26a)\\ \dot{\Phi} &= 3\,\frac{\cos \Phi}{\mathscr{R} \cdot \cosh \xi} + 2m \quad &(2.26b)\end{aligned}\right\}\sigma = +1$$

for the closed case $\sigma = -1$, (2.24),

$$\left.\begin{aligned} \dot{\xi} &= -3\frac{\cosh\xi \cdot \sin\Phi}{\mathscr{R}} && (2.27a) \\ \dot{\Phi} &= -3\frac{\cos\Phi}{\mathscr{R}\cdot\sinh\xi} + 2m && (2.27b) \end{aligned}\right\} \sigma = -1$$

and for the flat case $\sigma = 0$,

$$\left.\begin{aligned} \dot{\Phi} &= 2m && (2.28a) \\ {}^{(c)}\dot{\tilde{W}} &= -H\,{}^{(c)}\tilde{W} - 6({}^{(c)}\tilde{W})^2 \sin\Phi && (2.28b) \end{aligned}\right\} \sigma = 0$$

Obviously, the previous case (Sorg, 1992*a*, 1993) of an open universe ($\sigma = +1$) is again contained in (2.26) as the special situation $\xi \equiv 0, \Phi \equiv \chi$. Finally, we write down the corresponding Hamiltonian $\mathscr{H}_\mu$ deduced from the general cosmological ansatz (2.3) by considering the conditions (2.19)–(2.21):

$$\begin{aligned} {}^{(s)}\mathscr{H}_\mu \Rightarrow {}^{(w)}\mathscr{H}_\mu = \hbar c\Big\{ & 4\,{}^{(r)}W b_\mu (b^\lambda \gamma_\lambda) - \left({}^{(r)}W - \frac{m}{4}\right)\gamma_\mu \\ & + i\,{}^{(c)}\tilde{W}(4b_\mu b_\lambda - G_{\mu\gamma})\tilde{\gamma}^\lambda \\ & + i\,{}^{(c)}N\left(\frac{3}{2}b_\mu \cdot 1 - b^\lambda \Sigma_{\mu\lambda}\right)\Big\} \end{aligned} \qquad (2.29)$$

with the parametrization (2.23) for the open case ($\sigma = +1$) and with the parametrization (2.24) for the closed case ($\sigma = -1$).

Thus, we have arrived at the *two distinct* solutions (2.18) and (2.29) [in connection with (2.24)] for the *closed* universe, *one* solution (2.29) [in connection with (2.23)] for the *open* universe, and *one* solution (2.29) [in conjunction with (2.25)] for the flat universe. It is natural to expect that the physics becomes different for different Hamiltonians such as (2.18) versus (2.29), but that it will be similar for the same form of the Hamiltonian, even in topologically distinct universes ($\sigma = 0, \pm 1$). Now we are going to check this supposition by considering the physical densities in some detail.

3. PHYSICAL DENSITIES

Since the wave function ψ itself is regarded as unobservable, it is meaningful to consider certain gauge-invariant objects built up by the unobservable ψ and understood as the proper physical quantities. Among these we first have the *scalar density* ρ

$$\rho = \bar{\psi} \cdot \psi \qquad (3.1)$$

Next the *pseudo-density* $\tilde{\rho}$

$$\tilde{\rho} = \bar{\psi} \cdot \varepsilon \cdot \psi \tag{3.2}$$

Then the *current* j_μ

$$j_\mu = \bar{\psi} \cdot \gamma_\mu \cdot \psi \tag{3.3}$$

and its *axial* counterpart $\tilde{j}_\mu$,

$$\tilde{j}_\mu = i\bar{\psi} \cdot \bar{\gamma}_\mu \cdot \psi \qquad (\bar{\gamma}_\mu = \varepsilon \cdot \gamma_\mu) \tag{3.4}$$

And finally the *polarization* tensor $S_{\mu\nu}$,

$$S_{\mu\nu} = \frac{i}{2} \bar{\psi} \cdot \Sigma_{\mu\nu} \cdot \psi \tag{3.5}$$

Similarly to the operators involved (i.e., $\{1, \varepsilon, \gamma_\mu, \bar{\gamma}_\mu, \Sigma_{\mu\nu}\}$), the corresponding densities (3.1)–(3.5) are not quite independent, but are generally linked with each other by the following quadratic identities (Sorg, 1992*c*):

$$j^\mu j_\mu = -\tilde{j}^\mu \tilde{j}_\mu = \rho^2 + \tilde{\rho}^2 \tag{3.6a}$$

$$j^\mu \tilde{j}_\mu = 0 \tag{3.6b}$$

$$S_{\mu\nu} = \frac{1}{4} \frac{\tilde{\rho}}{\rho^2 + \tilde{\rho}^2} [j_\mu \tilde{j}_\nu - j_\nu \tilde{j}_\mu] - \frac{1}{4} \frac{\rho}{\rho^2 + \tilde{\rho}^2} \varepsilon_{\mu\nu\lambda\sigma} j^\lambda \tilde{j}^\sigma \tag{3.6c}$$

All other densities carried by the spinor field ψ can be composed of $\rho, \tilde{\rho}, j_\mu, \tilde{j}_\mu$ (and $S_{\mu\nu}$), such as the spin density $S_{\mu\nu\lambda}$,

$$S_{\mu\nu\lambda} = \bar{\psi} \cdot \{\Sigma_{\mu\nu}\gamma_\lambda + \gamma_\lambda \Sigma_{\mu\nu}\} \cdot \psi \tag{3.7}$$

which is the (Poincaré) dual of the axial current $\tilde{j}_\mu$, (3.4):

$$S_{\mu\nu\lambda} = \frac{1}{2} \varepsilon^\kappa{}_{\mu\nu\lambda} \tilde{j}_\kappa \tag{3.8}$$

A further example is the energy-momentum density ${}^{(D)}T_{\mu\nu}$, (2.7), which, for instance, is found for the first solution ${}^{(\sim)}\mathcal{H}_\mu$, (2.18), in its symmetrized form in terms of the basic densities (3.1)–(3.4) as

$$\begin{aligned} {}^{(D)}T_{(\mu\nu)} \Rightarrow {}^{(\sim)}T_{\mu\nu} &\equiv \frac{1}{2} \bar{\psi} \cdot \{\gamma_{(\mu}{}^{(\sim)}\mathcal{H}_{\nu)} + {}^{(\sim)}\bar{\mathcal{H}}_{(\nu}\gamma_{\mu)}\} \cdot \psi \\ &= Mc^2 \rho b_\mu b_\nu \pm 2 \frac{\hbar c}{\mathcal{R}} \left\{ b_{(\mu}\tilde{j}_{\nu)} - \frac{1}{4} (b^\lambda \tilde{j}_\lambda) G_{\mu\nu} \right\} \end{aligned} \tag{3.9}$$

Observe that this is just the more general form (2.8a) rather than the simple cosmological form (2.5). For the sake of completeness, we also write down the energy-momentum density for the second solution (2.29):

$$ {}^{(D)}T_{(\mu\nu)} \Rightarrow {}^{(w)}T_{\mu\nu} = \hbar c\left\{4\,{}^{(r)}W\rho b_\mu b_\nu - \left({}^{(r)}W - \frac{m}{4}\right)\rho G_{\mu\nu} - 8\,{}^{(c)}\tilde{W}(b_\mu b^\lambda\,{}^*S_{\lambda\nu} + b_\nu b^\lambda\,{}^*S_{\lambda\mu})\right\} \tag{3.10} $$

Contrary to the result (3.9), the energy-momentum density (3.10) can be brought into the strict cosmological form (2.5) by putting ${}^{(c)}\tilde{W} \Rightarrow 0$, but then the possibility of a closed universe ($\sigma = -1$) is eliminated; cf. (2.22). Due to this reason the *coupled* Dirac–Einstein equations exclude the *closed* universe (Mattes and Sorg, 1993)! It is directly evident from the result (3.9) [and also from (3.10) in the limiting case ${}^{(r)}W \Rightarrow m/4$] that the energy density $\mathcal{M}$ to be extracted from the tensor $T_{\mu\nu}$ is closely related to the scalar density ρ through

$$ \mathcal{M} \equiv T_{\mu\nu}b^\mu b^\nu \sim Mc^2\rho \qquad (\mathcal{R} \to \infty) \tag{3.11} $$

and therefore it is meaningful to look upon the quantity μ

$$ \mu := \rho\mathcal{R}^3 \tag{3.12} $$

quite generally as the "particle number," namely the matter energy contained in a comoving 3-cell of size $\mathcal{R}$ divided by the energy Mc^2 of a single particle. The evolution of that particle number μ during the expansion of the universe ($\mathcal{R} \to \infty$) is one of the main objects of interest in the next section.

Returning once more to the question of the interrelationship between the wave function ψ and the physical densities generated by it, one may ask whether that wave function ψ can be parametrized by some other variables which have a more direct physical meaning than the four complex components of the Dirac spinor themselves. Clearly, the physical information inherent in those densities (to be considered as being observable in principle) must already be contained in the unobservable wave function itself! A closer inspection of this problem has recently revealed (Mattes and Sorg, 1993) that a Dirac spinor ψ can always be parametrized by the scalar density ρ, the "intrinsic velocity" κ, the "spinor product" z (≤ 1), a "phase angle" χ, and an orthonormal tetrad $\tilde{\mathscr{E}}_\alpha = \{\tilde{\mathscr{E}}_{\alpha\mu}; \alpha = 0, \ldots, 3\} = \{b_\mu, \tilde{g}_\mu, \tilde{\pi}_\mu, \tilde{\lambda}_\mu\}$ such that

$$ \tilde{\mathscr{E}}_{\alpha\mu}\tilde{\mathscr{E}}_\beta{}^\mu = g_{\alpha\beta} \tag{3.13a} $$

$$ \tilde{\mathscr{E}}^\alpha{}_\mu\tilde{\mathscr{E}}_{\alpha\nu} = G_{\mu\nu} \tag{3.13b} $$

i.e.,

$$ b^\mu b_\mu = -\tilde{g}^\mu\tilde{g}_\mu = \cdots = +1 \tag{3.14a} $$

$$ b^\mu\tilde{g}_\mu = \tilde{\pi}^\mu\tilde{\lambda}_\mu = \cdots = 0 \tag{3.14b} $$

In terms of these new "physical parameters" the basic densities are expressed as follows:

$$\tilde{\rho} = z\rho \sinh 2\kappa \cdot \sin \chi \tag{3.15a}$$

$$j_\mu = \rho\{\cosh 2\kappa \cdot b_\mu + \sinh 2\kappa[\tilde{g}_\mu \cdot \cos \chi + \tilde{\pi}_\mu \cdot (1 - z^2)^{1/2} \sin \chi]\} \tag{3.15b}$$

$$\tilde{j}_\mu = -\rho[z \sinh 2\kappa \cdot \cos \chi \cdot b_\mu + z \cosh 2\kappa \cdot \tilde{g}_\mu + (1 - z^2)^{1/2} \cdot \tilde{\lambda}_\mu] \tag{3.15c}$$

$$S_{\mu\nu} = -\frac{1}{2} \rho \sinh 2\kappa[b_{[\mu}\tilde{g}_{\nu]} \cdot \sin \chi - b_{[\mu}\tilde{\pi}_{\nu]} \cdot (1 - z^2)^{1/2} \cos \chi]$$
$$+ \frac{1}{4} \rho b^\lambda \varepsilon_{\lambda\sigma\mu\nu}[z\tilde{g}^\sigma + \cosh 2\kappa \cdot (1 - z^2)^{1/2} \cdot \tilde{\lambda}^\sigma] \tag{3.15d}$$

There is a further physical density of great relevance: the polarization $M_{\mu\nu}$ defined through

$$M_{\mu\nu} = \frac{2e\hbar}{Mc} S_{\mu\nu} \tag{3.16}$$

Since this object characterizes both the electric and magnetic polarization of matter, it becomes necessary to specify some distinguished observer relative to whom the separation into the electric and magnetic parts of the polarization may be performed. Clearly, as this referee we will take an observer who is comoving with the Hubble flow b_μ. Then the polarization $M_{\mu\nu}$, (3.16), is split up into its electric and magnetic parts according to

$$M_{\mu\nu} = {}^{(e)}M_\mu b_\nu - {}^{(e)}M_\nu b_\mu + \varepsilon_{\mu\nu\lambda\sigma} b^{\lambda\ (m)}M^\sigma \tag{3.17}$$

with the electric dipole density ${}^{(e)}M_\mu$ given by

$${}^{(e)}M_\mu = \frac{e\hbar}{2Mc} \rho \sinh 2\kappa[\tilde{g}_\mu \cdot \sin \chi + \tilde{\pi}_\mu \cdot (1 - z^2)^{1/2} \cos \chi] \tag{3.18}$$

and its magnetic counterpart ${}^{(m)}M_\mu$ given by

$${}^{(m)}M_\mu = \frac{e\hbar}{2Mc} \rho[z \cdot \tilde{g}_\mu + (1 - z^2)^{1/2} \cosh 2\kappa \cdot \tilde{\lambda}_\mu] \tag{3.19}$$

Thus we have collected all the kinematical prerequisites in order to study in the next sections the physics in different universes ($\sigma = 0, \pm 1$) and ask questions like: What is the difference between the two closed universes [(2.18) and (2.29), $\sigma = -1$]? How does the polarization differ in the flat universe [(2.29), $\sigma = 0$] from that in the open universe [(2.29), $\sigma = +1$]? However, our main interest is in the question, which of the various universes is the most efficient one in order to produce as many "particles" μ as possible?

Evidently, these questions can be answered by investigating the appropriate densities, and it is not necessary to know the wave function ψ itself. Therefore one wants to have some method for directly computing the densities without resorting to the wave function. Such a method is available (Sorg, 1993) and is used extensively hereafter. It consists in writing down the first-order system for the densities, which may then be solved directly. To give a simple demonstration of this method, we consider the question of whether the energy density $\mathcal{M}$ is homogeneous for the closed universe (2.18). First, the gradient of the scalar density ρ is found by means of the relativistic Schrödinger equation (2.2) as

$$\begin{aligned} \partial_\mu \rho &= \frac{i}{\hbar c} \bar{\psi} \cdot [\bar{\mathcal{H}}_\mu - \mathcal{H}_\mu] \cdot \psi \\ &= 2\bar{\psi} \cdot \mathcal{L}_\mu \cdot \psi \\ &= -3\left(H\rho \pm \frac{1}{\mathcal{R}} \cdot \tilde{\rho} \right) b_\mu \end{aligned} \tag{3.20}$$

Consequently, this gradient points in the direction of the Hubble flow b_μ and therefore ρ *must be* homogeneous:

$$\mathcal{B}^\mu{}_\nu (\partial_\mu \rho) \equiv 0 \tag{3.21}$$

But the homogeneity of ρ is not sufficient for the homogeneity of the energy density ${}^{(\sim)}\mathcal{M}$ [cf. (3.9)]:

$${}^{(\sim)}\mathcal{M} = {}^{(\sim)}T_{\mu\nu} b^\mu b^\nu \tag{3.22a}$$

$$= Mc^2 \cdot \rho \pm \frac{3}{2} \frac{\hbar c}{\mathcal{R}} \tilde{I} \tag{3.22b}$$

$$\tilde{I} := b^\lambda \tilde{j}_\lambda \tag{3.22c}$$

Obviously, the homogeneity of ${}^{(\sim)}\mathcal{M}$ additionally requires the homogeneity of the scalar $\tilde{I}$ in (3.22c), whose gradient is readily written down as

$$\begin{aligned} \partial_\mu \tilde{I} &= (\nabla_\mu b^\lambda) \tilde{j}_\lambda + b^\lambda (\nabla_\mu \tilde{j}_\lambda) \\ &= H \mathcal{B}^\lambda{}_\mu \tilde{j}_\lambda + b^\lambda (\nabla_\mu \tilde{j}_\lambda) \end{aligned} \tag{3.23}$$

Here, the derivative of the axial current $\tilde{j}_\lambda$, (3.4), is computed in the following way:

$$\begin{aligned} \nabla_\mu \tilde{j}_\lambda &= \frac{1}{\hbar c} \bar{\psi} \cdot [\tilde{\gamma}_\lambda \cdot {}^{(\sim)}\mathcal{H}_\mu - {}^{(\sim)}\bar{\mathcal{H}}_\mu \cdot \tilde{\gamma}_\lambda] \cdot \psi \\ &= 2m\tilde{\rho} b_\mu b_\nu - 3H b_\mu \tilde{j}_\lambda - H b_\lambda \tilde{j}_\mu \\ &\quad + H\tilde{I} \cdot G_{\mu\lambda} \pm \frac{1}{\mathcal{R}} b^\sigma \tilde{j}^\rho \cdot \varepsilon_{\rho\mu\sigma\lambda} \end{aligned} \tag{3.24}$$

and consequently the desired gradient (3.23) becomes

$$\partial_\mu \tilde{I} = (2m\tilde{\rho} - 3H\tilde{I})b_\mu \tag{3.25}$$

which says that $\tilde{I}$ is indeed homogeneous. But with both scalars ρ of (3.20) and $\tilde{I}$ of (3.25) being homogeneous, the energy density ${}^{(\sim)}\mathcal{M}$ of (3.22) must be homogeneous, too, and we have our desired result.

By a similar argument, one readily shows that the pressure $\mathcal{P}$ and the energy-density $\mathcal{M}$ are really homogeneous for both types of energy-momentum densities (3.9) and (3.10) and therefore the requirements (2.6) are satisfied by our solutions (2.18) and (2.29) to the Dirac equation. This is our method for analyzing the physical densities in the various universes, mentioned above, and we shall now present their peculiar features.

4. COSMOLOGICAL PRINCIPLE AND POLARIZATION

Now we have accumulated enough material to face the puzzle of the cosmological principle, whose origin is assumed to range back to the global quantum state preceding the standard phase. Here we want to adopt the viewpoint that this quantum state may be adequately described by our present solutions to the Dirac equation in curved space-time. The point of interest is that spin necessarily breaks the isotropy and thus the *perfect* cosmological principle must be spoiled. However, for the RW symmetry of space-time we need the cosmological principle only with respect to the energy-momentum density $T_{\mu\nu}$ emerging in the Einstein equations (1.1). The other densities, e.g., the pseudoscalar $\tilde{\rho}$, may well be inhomogeneous. Thus, the question arises, which of the four solutions worked out so far comes closest to a homogeneous and isotropic density distribution for the early universe? Moreover, one wants to know whether there is some interplay of this question of highest symmetry with the topological type ($\sigma = 0, \pm 1$) of the universe. Clearly, it is very tempting to speculate that some of those primeval inhomogeneities or anisotropies due to polarization have possibly survived the phase transition into the standard phase and eventually may be observable even today as certain structures in the matter or microwave distribution.

4.1. Anisotropy and Polarization

First, we consider the closed universe solution (2.18) whose energy-momentum density ${}^{(\sim)}T_{\mu\nu}$ is given by equation (3.9). It is obvious that this density does not satisfy the strict cosmological principle (1.2), but differs from that cosmological form by some polarization term ${}^{(\sim)}\overset{*}{T}_{\mu\nu}$:

$${}^{(\sim)}T_{\mu\nu} = {}^{(\sim)}\overset{\circ}{T}_{\mu\nu} + {}^{(\sim)}\overset{*}{T}_{\mu\nu} \tag{4.1}$$

Here, the polarization part is given by

$$^{(\sim)}\overset{*}{T}_{\mu\nu} = \pm 2\frac{\hbar c}{\mathscr{R}}\, b_{(\mu}\mathscr{B}^{\lambda}{}_{\nu)}\tilde{j}_{\lambda} \tag{4.2a}$$

$$= \mp 4\frac{Mc^2}{e\mathscr{R}}\cosh 2\kappa \cdot b_{(\mu}\,{}^{(m)}M_{\nu)} \tag{4.2b}$$

but the cosmic part $^{(\sim)}\overset{\circ}{T}_{\mu\nu}$ exhibits the desired cosmological form (see end of Section 3):

$$^{(\sim)}\overset{\circ}{T}_{\mu\nu} = {}^{(\sim)}\mathscr{M} b_\mu b_\nu - {}^{(\sim)}\mathscr{P}\mathscr{B}_{\mu\nu} \tag{4.3a}$$

$$^{(\sim)}\mathscr{M} = Mc^2\rho + 3\cdot{}^{(\sim)}\mathscr{P} \tag{4.3b}$$

$$^{(\sim)}\mathscr{P} = \pm\frac{1}{2}\hbar c\frac{\tilde{I}}{\mathscr{R}} \tag{4.3c}$$

Amazingly enough, the cosmic part itself is also source-free:

$$\nabla^{\mu}\,{}^{(\sim)}\overset{\circ}{T}_{\mu\nu} = 0 \tag{4.4}$$

so that the work-energy theorem (1.3a) still applies, despite the presence of polarization. (Obviously, the polarization does not contribute to the energy density $\mathscr{M}$ nor to pressure $\mathscr{P}$.)

Thus, matter energy can be produced on account of the negative pressure, but the total amount of produced matter in the limit $\mathscr{R}\to\infty$ is strictly limited (see next section). For the moment, we are mainly interested in the behavior of the polarization term because it apparently spoils the exactness of the cosmological principle. However, observe that the axial current $\tilde{j}_\nu$ in the polarization part (4.2) contains the scalar density ρ according to equation (3.15c) and this density regulates the fading out of energy density $^{(\sim)}\mathscr{M}$, (4.3b), and pressure $^{(\sim)}\mathscr{P}$, (4.3c), with increasing size $\mathscr{R}$ of the universe. But due to the emergence of an additional inverse radius $\mathscr{R}^{-1}$, the polarization term (4.2) dies out relative to the energy density $^{(\sim)}\mathscr{M}$ for increasing $\mathscr{R}$, and the strict cosmological shape (1.2) with $\mathscr{P}=0$ is approximated more and more. In this way, the cosmological principle may be valid to a high degree in that instant when the transition to the standard phase occurs, which then starts with the initial condition of homogeneity and isotropy of matter distribution. It seems to us that this mechanism provides a satisfactory explanation for the old puzzle of the origin of the cosmological principle, because it furthermore signals some possible generalizations: one may suppose that the arguments will remain qualitatively valid if one considers *both* a more unsymmetric set of solutions to the Dirac equation *and* a non-RW background. In such a situation, we will expect that the growing size of the universe damps the unsymmetric constituents of the spinor energy-momentum density and leaves it in the cosmological form (1.2), which then enforces the RW symmetry of the space-time geometry via the Einstein equation (1.1).

4.2. Decay of Polarization

Is this damping into the RW symmetry a specific feature exclusively of the *closed* universe carrying the solution (2.18)? To answer this question, we have to consider the alternative solution ${}^{(w)}\mathcal{H}_\mu$ of (2.29) with the corresponding energy-momentum density ${}^{(w)}T_{\mu\nu}$, (3.10). This density may be split up in an analogous way as in the preceding case, (4.1), namely

$$ {}^{(w)}T_{\mu\nu} = {}^{(w)}\mathring{T}_{\mu\nu} + {}^{(w)}\overset{*}{T}_{\mu\nu} \tag{4.5} $$

where the cosmic part ${}^{(w)}\mathring{T}_{\mu\nu}$ has the desired form,

$$ {}^{(w)}\mathring{T}_{\mu\nu} = {}^{(w)}\mathcal{M} b_\mu b_\nu - {}^{(w)}\mathcal{P}\mathcal{B}_{\mu\nu} \tag{4.6a} $$

$$ {}^{(w)}\mathcal{M} = 3\hbar c\rho\left({}^{(r)}W + \frac{m}{12}\right) \tag{4.6b} $$

$$ {}^{(w)}\mathcal{P} = \hbar c\rho\left({}^{(r)}W - \frac{m}{4}\right) \tag{4.6c} $$

and the polarization part ${}^{(w)}\overset{*}{T}_{\mu\nu}$ is found as

$$ {}^{(w)}\overset{*}{T}_{\mu\nu} = -8\,{}^{(c)}\tilde{W}\{b_\mu b^\lambda\,{}^*S_{\lambda\nu} + b_\nu b^\lambda\,{}^*S_{\lambda\mu}\} \tag{4.7a} $$

$$ = 8\,\frac{Mc^2}{e}\,{}^{(c)}\tilde{W} b_{(\mu}\,{}^{(m)}M_{\nu)} \tag{4.7b} $$

The cosmic part ${}^{(w)}\mathring{T}_{\mu\nu}$, (4.6a), has again vanishing source

$$ \nabla^\mu\,{}^{(w)}\mathring{T}_{\mu\nu} = 0 \tag{4.8} $$

(and therefore also the polarization term), so that the work-energy theorem (1.3a) also applies strictly in the present case! Thus, the polarization has no effect upon matter production.

Now the interesting question in this case is again whether the polarization part (4.7) vanishes more rapidly (for increasing radius $\mathcal{R}$) than the cosmic part (4.6) so that the damping mechanism into the RW symmetry can occur again. Obviously this would be true if one could show that the scalar ${}^{(c)}\tilde{W}$ decreases for $\mathcal{R} \to \infty$, and if it decreases like $\mathcal{R}^{-1}$, the damping to RW symmetry would be as strong as in the preceding case (4.2) for the closed universe. Now, the equation of motion for ${}^{(c)}\tilde{W}$ is deduced from (2.11b) as

$$ \frac{d}{d\theta}(\mathcal{R}\cdot{}^{(c)}\tilde{W}) = -3({}^{(c)}N + H)(\mathcal{R}\cdot{}^{(c)}\tilde{W}) \tag{4.9} $$

and this equation is most conveniently solved by use of the parametrizations (2.23)–(2.25). For the open case (2.23) we find for the variable $\xi(\theta)$

$$\sinh \xi = \sinh \xi_{\text{in}} \cdot \frac{1 - T_+^2(\theta)}{1 + 2 \cosh \xi_{\text{in}} \cdot T_+(\theta) + T_+^2(\theta)} \tag{4.10a}$$

with the time function $T_+(\theta)$ given by

$$T_+(\theta) := \tanh\left[\frac{3}{2} \int_{\theta_{\text{in}}}^{\theta} \frac{\sin \Phi}{\mathscr{R}}\, d\theta\right] \tag{4.10b}$$

Similarly, for the closed case (2.24) we obtain

$$\cosh \xi = \cosh \xi_{\text{in}} \cdot \frac{1 + T_-^2(\theta)}{1 + 2 \sinh \xi_{\text{in}} \cdot T_-(\theta) - T_-^2(\theta)} \tag{4.11a}$$

where the time function $T_-(\theta)$ is now given by

$$T_-(\theta) := \tan\left[\frac{3}{2} \int_{\theta_{\text{in}}}^{\theta} \frac{\sin \Phi}{\mathscr{R}}\, d\theta\right] \tag{4.11b}$$

Finally, the flat case (2.25) directly yields for ${}^{(c)}\tilde{W}$

$$\mathscr{R} \cdot {}^{(c)}\tilde{W} = \frac{\mathscr{R}_{\text{in}} \cdot {}^{(c)}\tilde{W}_{\text{in}}}{1 + \mathscr{R}_{\text{in}} \cdot {}^{(c)}\tilde{W}_{\text{in}} \cdot T_0(\theta)} \tag{4.12a}$$

with the time function T_0:

$$T_0(\theta) := \int_{\theta_{\text{in}}}^{\theta} \frac{\sin \Phi}{\mathscr{R}}\, d\theta \tag{4.12b}$$

These results clearly show that whenever the time functions T_+, T_-, T_0 are of bounded variation for $\mathscr{R} \to \infty$, the scalar ${}^{(c)}\tilde{W}$ vanishes like $\mathscr{R}^{-1}$, and consequently the polarization terms ${}^{(w)}\overset{*}{T}_{\mu\nu}$ of (4.7) must die out as rapidly as in the former case (4.2)! Thus, we arrive at the result that the polarization is damped out in all three types of RW universe ($\sigma = 0, \pm 1$) and therefore there is no preference for one of these topologies. However, if one wants to insist on the *strict* cosmological principle ($\rightsquigarrow$ vanishing $\overset{*}{T}_{\mu\nu}$), there is left only the *open* universe ($\sigma = +1$) (Mattes and Sorg, 1993).

4.3. Polarization Catastrophe

Concerning the primeval fate of the universe, there are certain critical points which could have led to its premature death. The first danger consists in an early recollapse ($\mathscr{R} \to 0$) after a short lifetime [this problem is treated in Ochs and Sorg (1993)]. The second danger is an anisotropic collapse ($\rightsquigarrow$ lower-dimensional collapse) which emerges in our present context as a kind of polarization catastrophe (${}^{(c)}\tilde{W} \to \infty$). In order to avoid this, the denominators on the right-hand sides of all three cases (4.10a), (4.11a), and (4.12a) must always remain nonzero, which imposes some restrictions upon the expansion law $\mathcal{R} = \mathcal{R}(\theta)$. For instance, in the simplest case (4.12a) we must require for all times θ

$$|T_0(\theta)| < |\mathcal{R}_{\text{in}} \cdot {}^{(c)}\tilde{W}_{\text{in}}|^{-1} \tag{4.13}$$

in order that a polarization catastrophe (${}^{(c)}\tilde{W} = \infty$) be avoided. Similar conclusions can be drawn for $T_{\pm}(\theta)$ from the other two situations (4.10a) and (4.11a). Thus, from the viewpoint of the anthropic principle, the polarization catastrophe acts as a sort of Darwinistic selection principle: if the primeval universe is not homogeneous and isotropic enough in order to sufficiently approximate the RW symmetry during its further expansion ($\mathcal{R} \to \infty$), it must undergo self-destruction. Observe, however, that this effect refers exclusively to the second type of solution (2.29); the first type (2.18) always excludes such a catastrophe.

4.4. Almost Exact Cosmological Principle

The first type (2.18) is also interesting from another point of view: As was demonstrated through the arguments at the end of the preceding section, the scalars $\rho, \tilde{\rho}, I, \tilde{I}$ are strictly homogeneous for the solution (2.18). Thus, the general shape (3.15) of the densities says that the physical parameters z, κ, χ must also be homogeneous. It is evident that this situation reaches to the cosmological principle as close as possible. It must be clear that the very phenomenon of spin must break isotropy necessarily, but homogeneity can be perfect, and this is achieved just by our solution (2.18). It is readily seen that the second kind (2.29) of our solutions cannot exhibit such a high degree of homogeneity. By means of the method mentioned above we find for the derivative of the intrinsic velocity κ

$$\begin{aligned} 2 \cdot \partial_\mu \kappa = \tilde{g}_\mu &\left\{({}^{(c)}N + H) \cdot \cos\chi - 2\left({}^{(r)}W - \frac{m}{4}\right) \cdot \sin\chi\right\} \\ &+ \tilde{\pi}_\mu \cdot (1 - z^2)^{1/2}\left\{({}^{(c)}N + H) \cdot \sin\chi + 2\left({}^{(r)}W - \frac{m}{4}\right) \cdot \cos\chi\right\} \\ &- 6z\, {}^{(c)}\tilde{W} \cdot \sin\chi\, b_\mu \end{aligned} \tag{4.14}$$

Thus, the intrinsic velocity κ will in general neither be homogeneous nor time independent, except in the special case of a flat universe ($\sigma = 0$) with vanishing scalar field ${}^{(c)}\tilde{W}$; cf. (2.25).

A similar result also holds for the spinor product z:

$$\begin{aligned}\sinh 2\kappa \cdot (\partial_\mu z) = \tilde{\lambda}_\mu (1 - z^2)^{1/2} \left\{ ({}^{(c)}N + H) \cdot \cos \chi - 2\left({}^{(r)}W - \frac{m}{4} \right) \cdot \sin \chi \right\} \\ - \tilde{\pi}_\mu \cdot z(1 - z^2)^{1/2} \cosh 2\kappa \left\{ ({}^{(c)}N + H) \cdot \sin \chi \right. \\ \left. + 2\left({}^{(r)}W - \frac{m}{4} \right) \cdot \cos \chi \right\} + 2\, {}^{(c)}\tilde{W} (1 - z^2)^{1/2} \sinh 2\kappa \cdot \tilde{\pi}_\mu \\ - 6\, {}^{(c)}\tilde{W} \cosh 2\kappa \sin \chi\, (1 - z^2) b_\mu \qquad (4.15)\end{aligned}$$

Here again, the physical parameter z *cannot* be homogeneous in an open or closed universe, [cf. (2.23) and (2.24)], but it can be homogeneous in the flat universe (2.25), namely in the special case of vanishing ${}^{(c)}\tilde{W}$.

However, the rapid phase angle χ may well be homogeneous, provided ${}^{(c)}\tilde{W}$ vanishes permanently:

$$\begin{aligned}(\partial_\mu \chi) = -\tilde{g}_\mu \cdot \coth 2\kappa \left\{ ({}^{(c)}N + H) \cdot \sin \chi \right. \\ \left. + 2\left({}^{(r)}W - \frac{m}{4} \right) \cdot \cos \chi - 2\, {}^{(c)}\tilde{W} \cdot \sinh 2\kappa \right\} \\ - \tilde{\lambda}_\mu \frac{(1 - z^2)^{1/2}}{z \sinh 2\kappa} \left\{ ({}^{(c)}N + H) \cdot \sin \chi + 2\left({}^{(r)}W - \frac{m}{4} \right) \cdot \cos \chi \right\} \\ + 6\left({}^{(r)}W + \frac{m}{12} \right) b_\mu - 6 \frac{{}^{(c)}\tilde{W}}{z} \coth 2\kappa \cos \chi\, b_\mu \qquad (4.16)\end{aligned}$$

Thus, for achieving homogeneity of χ we merely have to require $\Phi = \chi + \pi/2$, in addition to ${}^{(c)}\tilde{W} \equiv 0$, and then we find the simplified dynamical equation for Φ:

$$\dot{\Phi} = 2m + 3 \frac{\cos \Phi}{\mathcal{R}} \qquad (4.17)$$

in agreement with previous results (Sorg, 1993). (Here, the phase shift $\zeta = \pi/2$ has been absorbed into the rapid variable χ.)

Thus, we arrive at the result that the open and closed universes carrying the solution (2.29) are less homogeneous than the closed universe (2.18); and it is only the *flat* universe within the set (2.29) which can achieve the high degree of homogeneity like that of the closed type (2.18)!

4.5. Rotation of the Triad

As is evident from the physical densities (3.15), the orthonormal triad $\{\tilde{\mathscr{E}}_{i\mu}\} = \{\tilde{g}_\mu, \tilde{\pi}_\mu, \tilde{\lambda}_\mu\}$ defines a reference frame relative to which the densities are oscillating, and the corresponding frequency is approximately on the Compton scale ($\omega_c \approx 2mc = 2Mc^2/\hbar$); cf. (4.17) for $\mathscr{R} \gg m$. However, if one is interested in the effective time behavior of the densities relative to a comoving observer, one must also take account of the rotation of the triad relative to such an observer. As we shall readily see, the triad rotation is also governed by the scalar ${}^{(c)}\tilde{W}$ and therefore the rotation dies out with this scalar field for $\mathscr{R} \to \infty$ in a similar way as the anisotropy and inhomogeneity effects mentioned above. In order to see this clearly, we take the time derivative $b^\mu \nabla_\mu$ of the triad vectors $\tilde{\mathscr{E}}_{i\nu}$ and find by means of our method described above

$$b^\mu \nabla_\mu \tilde{\mathscr{E}}_{i\nu} = \tilde{\mathscr{E}}_{j\nu} \tilde{\omega}^j{}_i \tag{4.18}$$

or if we prefer to work with the rotation 3-vector $\tilde{\omega}_i$ (in place of the rotation matrix $\tilde{\omega}^i{}_j$)

$$b^\mu \nabla_\mu \tilde{\mathscr{E}}_{i\nu} = \varepsilon_i{}^{jk} \tilde{\omega}_j \tilde{\mathscr{E}}_{k\nu} \tag{4.19}$$

Here, the rotation vector (matrix) is given through

$$\tilde{\omega}^1{}_2 = \tilde{\omega}_3 = -6\,{}^{(c)}\tilde{W} \cdot \frac{(1-z^2)^{1/2}}{z} \cdot \coth 2\kappa \cdot \cos\chi \tag{4.20a}$$

$$\tilde{\omega}^2{}_3 = \tilde{\omega}_1 = -6\,{}^{(c)}\tilde{W} \frac{\cos\chi}{\sinh 2\kappa} \tag{4.20b}$$

$$\tilde{\omega}^3{}_1 = \tilde{\omega}_2 = -6\,{}^{(c)}\tilde{W} \cdot (1-z^2)^{1/2} \frac{\sin\chi}{\sinh 2\kappa} \tag{4.20c}$$

Observe that in the general case, neither of the scalars κ, χ, z is homogeneous [(4.14)–(4.16)] and consequently the rotation vector $\{\tilde{\omega}_i\}$ is both space and time dependent. Clearly, this result yields a very complicated kinematical behavior of the currents $j_\mu, \tilde{j}_\mu$ and the polarization $S_{\mu\nu}$. For this reason we shall henceforth restrict ourselves to the case ${}^{(c)}\tilde{W} \equiv 0$ and shall now work out under this presumption the different physics occurring in the three different universes ($\sigma = 0, \pm 1$).

5. CLOSED UNIVERSE ($\sigma = -1$), TYPE (2.18)

It is clear that the simplest universe is given by the case (2.18) because there are no intrinsic degrees of freedom for matter, such as the variables ξ and χ in the more complicated situation (2.29). First, we want to see how the orthonormal triad $\tilde{\mathscr{E}}_{i\mu}$ ($i = 1, 2, 3$) [cf. (3.13–(3.14)] looks in the present case because this triad essentially determines the polarization $M_{\mu\nu}$, the flow of the current j_μ, etc. Through the method described at the end of Section 3, we readily find the following field equations for the triad:

$$''\nabla_\nu \tilde{g}_\lambda = \pm\frac{1}{\mathscr{R}}[\tilde{\lambda}_\lambda \tilde{\pi}_\nu - \tilde{\lambda}_\nu \tilde{\pi}_\lambda] \tag{5.1a}$$

$$''\nabla_\nu \tilde{\lambda}_\lambda = \pm\frac{1}{\mathscr{R}}[\tilde{\pi}_\lambda \tilde{g}_\nu - \tilde{\pi}_\nu \tilde{g}_\lambda] \tag{5.1b}$$

$$''\nabla_\nu \tilde{\pi}_\lambda = \pm\frac{1}{\mathscr{R}}[\tilde{g}_\lambda \tilde{\lambda}_\nu - \tilde{g}_\nu \tilde{\lambda}_\lambda] \tag{5.1c}$$

or in compact notation

$$''\nabla_\nu \tilde{\mathscr{E}}_{i\lambda} = \tilde{\mathscr{E}}_{j\lambda} \tilde{\Omega}^j{}_{i\nu} \tag{5.2}$$

with the rotation matrix $\tilde{\Omega}$ given by

$$\tilde{\Omega}^j{}_{i\nu} = \pm\frac{1}{\mathscr{R}}\epsilon^j{}_{ik}\tilde{\mathscr{E}}^k{}_\nu \tag{5.3}$$

The derivative $''\nabla$ occurring here is the surface derivative (Sorg, 1993), i.e. $''\nabla = \mathscr{B} \circ \nabla \circ \mathscr{B}$, which is induced by the original derivative ∇ on the 3-distribution formed by the triad field $\{\tilde{g}_\mu, \tilde{\lambda}_\mu, \tilde{\pi}_\mu\}$.

The triad (5.1) has some interesting properties to be discussed shortly, but first let us check the consistency of their field equations (5.1), (5.2) by differentiating once more across the 3-distribution ($\hat{\nabla}_\mu := \mathscr{B}^\nu{}_\mu {}''\nabla_\nu$):

$$[\hat{\nabla}_\mu \hat{\nabla}_\nu - \hat{\nabla}_\nu \hat{\nabla}_\mu]\tilde{\mathscr{E}}_{i\lambda} = \left(\frac{1}{\mathscr{R}}\right)^2 [\tilde{\mathscr{E}}_{i\mu}\mathscr{B}_{\lambda\nu} - \tilde{\mathscr{E}}_{i\nu}\mathscr{B}_{\lambda\mu}] \tag{5.4}$$

Comparing this to the identity

$$[\hat{\nabla}_\mu \hat{\nabla}_\nu - \hat{\nabla}_\nu \hat{\nabla}_\mu]\tilde{\mathscr{E}}_{i\lambda} = -\hat{R}^\sigma{}_{\lambda\mu\nu}\tilde{\mathscr{E}}_{i\sigma} \tag{5.5}$$

where $\hat{R}$ is the Riemannian of the integral surface of the 3-distribution, readily yields for that curvature tensor

$$\hat{R}_{\sigma\lambda\mu\nu} = \left(\frac{1}{\mathscr{R}}\right)^2 [\mathscr{B}_{\lambda\mu}\mathscr{B}_{\sigma\nu} - \mathscr{B}_{\sigma\mu}\mathscr{B}_{\lambda\nu}] \tag{5.6}$$

which is just the expected result for a homogeneous isotropic 3-surface (observe $\mathscr{B}_{\mu\nu} \equiv \tilde{\mathscr{E}}^i{}_\mu \tilde{\mathscr{E}}_{i\nu}$)! Next, notice that any one of the triad vectors (5.2) is parallel transported along its own integral curve, e.g.,

$$\tilde{g}^\nu(\hat{\nabla}_\nu \tilde{g}_\lambda) = 0, \quad \text{etc.} \tag{5.7}$$

but nevertheless the triad is unable to establish a 3-coordinate system because the Frobenius integrability condition is not satisfied:

$$[\hat{\nabla}_\nu \tilde{g}_\lambda - \hat{\nabla}_\lambda \tilde{g}_\nu] \not\propto [\tilde{h}_\nu \tilde{g}_\lambda - \tilde{h}_\lambda \tilde{g}_\nu] \tag{5.8}$$

(with an arbitrary 1-form $\tilde{h}_\nu$). On the other hand, the triad is indeed a *global* 3-frame over the 3-dimensional sphere S^3, as may be checked in the following way: forming some subframe $\tilde{e} = \{\tilde{e}_a; a = 1, 2\}$, e.g., $\tilde{e}_{1\mu} = \tilde{\lambda}_\mu$, $\tilde{e}_{2\mu} = \tilde{\pi}_\mu$, and transporting this 2-frame parallel along the third triad vector $\tilde{g}_\mu$ yields

$$\tilde{g}^\nu(\hat{\nabla}_\nu \tilde{e}_a) = \tilde{e}_b \overset{*}{\Omega}{}^b{}_a \tag{5.9}$$

with the subrotation matrix $\overset{*}{\Omega}$ being found as the generator of the rotation subgroup $\mathcal{SO}(2)$:

$$\overset{*}{\Omega}{}^b{}_a = \pm \frac{1}{\mathcal{R}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{5.10}$$

Thus the solution of (5.9) for the 2-frame $\tilde{e}_a$ in terms of the proper length s along the geodesic line with tangent vector $\tilde{g}_\nu$ is

$$\tilde{e}(s) = \tilde{e}(0) \cdot \exp[s\overset{*}{\Omega}] \tag{5.11}$$

with the $\mathcal{SO}(2)$ group element being given by

$$\exp[s\overset{*}{\Omega}] = \cos\left(\frac{s}{\mathcal{R}}\right) \pm \sin\left(\frac{s}{\mathcal{R}}\right) \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{5.12}$$

Thus following that geodesic line around the whole universe, the $\mathcal{SO}(2)$ group element (5.12) becomes at the end ($s = 2\pi\mathcal{R}$) of the circumference

$$\exp[2\pi R\overset{*}{\Omega}] = 1 \tag{5.13}$$

and the 2-frame consistently ends up with its starting configuration at $s = 0$.

After the triad $\tilde{\mathcal{E}}$, (3.13)–(3.14), is known, a glimpse at the physical densities (3.15) shows that we have to compute the equations of motion for the physical parameters z, κ, and χ. As is easily verified, the first scalar is very simple, namely

$$z \equiv 1 \tag{5.14}$$

whereas the other two obey a coupled first-order system:

$$b^\mu \partial_\mu \chi \doteqdot \dot{\chi} = 2m \pm 3 \frac{\cos \chi}{\mathcal{R} \sinh 2\kappa} \tag{5.15a}$$

$$\dot{\kappa} = \pm \frac{3}{2} \frac{\cosh 2\kappa \cdot \sin \chi}{\mathcal{R}} \tag{5.15b}$$

From this result we conclude that the angle χ increases roughly in the order of magnitude of the first term on the right-hand side of (5.15a), which gives a certain oscillatory character to the physical densities. For instance, the current j_μ of (3.15b) is oscillating linearly

$$j_\mu = \rho\{\cosh 2\kappa \cdot b_\mu + \sinh 2\kappa \cdot \cos\chi \cdot \tilde{g}_\mu\} \tag{5.16}$$

which is a manifestation of the well-known *Schrödinger trembling* (Mattes and Sorg, 1993). Furthermore, the dipole densities (3.18) + (3.19) are found as

$$^{(e)}M_\mu = \frac{e\hbar}{2Mc}\rho \sinh 2\kappa \cdot \sin\chi \cdot \tilde{g}_\mu \tag{5.17a}$$

$$^{(m)}M_\mu = \frac{e\hbar}{2Mc}\rho \tilde{g}_\mu \tag{5.17b}$$

and clearly exhibit the qualitatively quite different character of both densities: whereas the electric component (5.17a) is always rapidly oscillating, the magnetic counterpart (5.17b) does not contain any rapidly varying constituent (whenever the universe has become large enough)! Observe here that the result (3.20) says for this large-size limit

$$\dot{\rho} \sim -3H\rho \tag{5.18}$$

i.e., the scalar density drops down softly with the radius $\mathcal{R}$ as

$$\rho \sim \frac{\text{const}}{\mathcal{R}^3} \tag{5.19}$$

This implies that any comoving 3-cell carries a *constant* magnetic polarization in the direction of $\tilde{g}_\mu$ [$\rightsquigarrow$ longitudinal polarization; cf. (5.16)]!

However, the most interesting feature of such a universe is its energy production via the pumping mechanism (1.3a). As may be read off from the energy-density $^{(\sim)}\mathcal{M}$ in (3.22), the matter energy in a comoving 3-cell is measured adequately by the "particle number" μ, (3.12), in the limit of infinite size ($\mathcal{R} \to \infty$) of the universe. Therefore it is meaningful to set up an equation of motion for the variable μ, which is readily deduced from equation (3.20) as

$$\dot{\mu} = \mp 3\frac{\tilde{\mu}}{\mathcal{R}} \tag{5.20}$$

where $\tilde{\mu}$ has been defined analogously to the particle number μ:

$$\tilde{\mu} := \tilde{\rho}\mathcal{R}^3 \tag{5.21}$$

But since μ couples to $\tilde{\mu}$, we have to establish also the equation of motion for the latter variable, which, however, is readily obtained by the same technique and is found as

$$\dot{\tilde{\mu}} = \pm 3\frac{\mu}{\mathcal{R}} - 2m\tilde{v} \tag{5.22}$$

with the scalar $\tilde{\nu}$ defined as

$$\tilde{\nu} := \tilde{I}\mathscr{R}^3 \tag{5.23}$$

[for $\tilde{I}$ see (3.22c)]. Unfortunately, the second variable $\tilde{\mu}$ couples now to a third scalar $\tilde{\nu}$, and we have also to add the equation of motion for the latter one in order to close our system:

$$\dot{\tilde{\nu}} = 2m\tilde{\mu} \tag{5.24}$$

But now we have the complete system (5.20), (5.22), (5.24) and thus are well prepared to discuss the corresponding energy production, which is regulated by the conservation law

$$\frac{d}{d\theta}\{\mu^2 + \tilde{\mu}^2 + \tilde{\nu}^2\} = 0 \tag{5.25}$$

to be deduced immediately from that system. This important result says that the particle number μ is strictly bounded, independently of the specific way of expansion of the universe. Thus, *it is impossible to generate an appreciable amount of matter by cosmic pumping in the closed universe of the type* (2.18).

6. FLAT UNIVERSE, $\sigma = 0$

This universe carries a very simple physics, because with the constraints (2.25) and vanishing scalar ${}^{(c)}\tilde{W}$ we arrive at

$$ {}^{(c)}N + H = 0 \tag{6.1a}$$

$$ {}^{(r)}W = \frac{m}{4} \tag{6.1b}$$

so that the Hamiltonian ${}^{(w)}\mathscr{H}_\mu$ in (2.29) is simplified into

$$ {}^{(w)}\mathscr{H}_\mu \Rightarrow {}^{(*)}\mathscr{H}_\mu = Mc^2 b_\mu(b^\lambda \gamma_\lambda) - i\hbar c H\left(\frac{3}{2}b_\mu \cdot 1 - b^\lambda \Sigma_{\mu\lambda}\right) \tag{6.2}$$

As a consequence, the energy-momentum density ${}^{(w)}T_{\mu\nu}$, (3.10), adopts a very simple form, namely

$$ {}^{(w)}T_{\mu\nu} \Rightarrow {}^{(*)}T_{\mu\nu} = Mc^2 \rho b_\mu b_\nu \tag{6.3}$$

which says that the pressure $\mathscr{P}$ must always vanish in a *flat* universe! But in this case, there is no energy production via the mechanism (1.3a) and the matter energy in any comoving 3-cell must be constant! Thus, *in a flat universe there never occurs homogeneous matter production*:

$$\mu \Rightarrow \mu_* = \text{const} \tag{6.4}$$

contrary to the closed universe [cf. (5.20)]!

Through this result, one gets the impression that the flat universe is the most boring one of all three types. Indeed, this supposition is verified readily by writing down the field equations for the currents according to the method described above [cf. (3.20)]:

$$\nabla_\mu j_\nu = 8mb_\mu b^\lambda S_{\lambda\nu} - 3Hb_\mu j_\nu - H[j_\mu b_\nu - G_{\mu\nu}(b^\lambda j_\lambda)] \tag{6.5a}$$

$$\nabla_\mu \tilde{j}_\nu = 2mb_\mu b_\nu - 3Hb_\mu \tilde{j}_\nu - H[b_\nu \tilde{j}_\mu - G_{\mu\nu}(b^\lambda \tilde{j}_\lambda)] \tag{6.5b}$$

Introducing here the densities (3.15), one arrives at the result that the triad is covariantly constant:

$$''\nabla_\mu \tilde{g}_\nu = ''\nabla_\mu \tilde{\pi}_\nu = ''\nabla_\mu \tilde{\lambda}_\nu \equiv 0 \tag{6.6}$$

Further, the spinor product z and the intrinsic velocity κ are both space-time independent, and finally the rapid phase angle χ obeys the very simple law

$$\dot{\chi} = 2m \tag{6.7}$$

as opposed to the case of the closed universe [cf. (5.15a)]. Clearly, the flat space-time ($\sigma = 0, H = 0$) is a subcase hereof with the scalar density ρ becoming space-time independent. However, it must be remarked that the flat universe is interesting insofar as it is the only case in which the Schrödinger trembling motion (Mattes and Sorg, 1993) can be suppressed, i.e., $\kappa = 0$, which implies

$$j_\mu = \rho b_\mu \tag{6.8}$$

and the Dirac current becomes proportional to the Hubble flow. Observe here that the form (6.8) for the current is the only possible one in order to obey the exact cosmological principle. Therefore, if one likes the argument that the universe must have been born in the highest symmetric state and was subsequently filled with Dirac matter through cosmic pumping, then our universe must be flat (Sorg, 1992*b*).

7. OPEN UNIVERSE, $\sigma = +1$

For the sake of comparison, we briefly discuss here the results for the open universe (Ochs and Sorg, 1994; Mattes *et al.*, 1993), which is the most interesting one with respect to matter production. Indeed, the equation of motion for the "particle number" μ is (for ${}^{(c)}\tilde{W} = 0$)

$$\dot{\mu} = 3({}^{(c)}N + H)\mu = 3\,\frac{\sin\chi}{\mathcal{R}}\,\mu \tag{7.1}$$

Actually, the equation of motion (7.1) does not put any limit upon the "particle number" μ; but as explained in connection with the work-energy theorem (1.3a), the energy production can effectively take place only as long as the pressure $\mathscr{P}$ is sufficiently negative, whereas the energy-momentum density ${}^{(w)}T_{\mu\nu}$, (3.10), does admit this only for a small enough radius $\mathscr{R}$ and for $\pi/2 < \chi < 3\pi/2$:

$$
\begin{aligned}
{}^{(w)}\mathscr{P} &= \hbar c \rho \left({}^{(r)}W - \frac{m}{4} \right) \\
&= \hbar c \rho \frac{\cos \chi}{2\mathscr{R}}
\end{aligned}
\tag{7.2}
$$

Thus, a rapidly growing universe ($\mathscr{R} \to \infty$) soon stops a further increase of the "particle number" μ, which is also easily recognized from its dynamical equation (7.1). Therefore, if one wants to have a further increase in energy, one must let the universe recollapse to such a small radius $\mathscr{R}$ *and* simultaneously χ must be just in the favorable phase interval mentioned above so that μ can experience a further increase! Such a matching of radius $\mathscr{R}$ and the phase angle χ, obeying [cf. (4.17)]

$$
\dot{\chi} = 3 \frac{\cos \chi}{\mathscr{R}} + 2m \tag{7.3}
$$

is possible either by considering the coupled Dirac–Einstein equations with a negative cosmological constant (Mattes *et al.*, 1993) or by arbitrarily imposing a convenient time dependence of the radius $\mathscr{R} = \mathscr{R}(\theta)$. As a special example of the latter method we choose (putting $r = m\mathscr{R}$, $t = m\theta$, $\dot{r} = dr/dt$)

$$
r(t) = t \left\{ 1 - r_* \left[1 - \tanh\left(t - \frac{\tau_1}{t} \right) \right] \sin^2\left(\frac{t}{\tau_2} \right) \right\} \tag{7.4}
$$

(see Fig. 1). By optimizing here the parameters r_*, τ_1, τ_2 we can get particle numbers as large as we want (Fig. 2).

Apart from its capability of unlimited energy production, the open universe is interesting also in some other respects. First of all, observe that some of the scalars, homogeneous in the closed and flat universes ($\sigma = 0, -1$), *must* now be inhomogeneous in the open case ($\sigma = +1$). Clearly, the scalar density ρ is still homogeneous:

$$
\partial_\mu \rho = 3^{(c)} N \rho b_\mu = 3\rho \left\{ \frac{\sin \chi}{\mathscr{R}} - H \right\} b_\mu \tag{7.5}
$$

[substitute ${}^{(w)}\mathscr{H}_\mu$, (2.29), with ${}^{(c)}\tilde{W} = 0$ for the general $\mathscr{H}_\mu$ in equation (3.20)!], but if we look at the "charge density" $I = b^\lambda j_\lambda = \rho \cosh 2\kappa$ we find

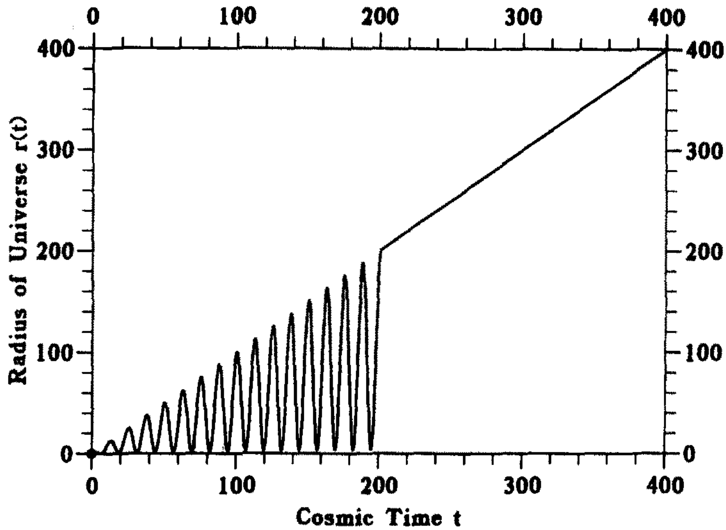

Fig. 1. Oscillating radius of the universe. By choosing the period τ_2 so that the phase condition $\pi/2 < \chi < 3\pi/2$ is matched optimally, one can obtain an unlimited particle number μ. After the end of the oscillations ($t \approx 200$), a phase transition is assumed to occur, after which the universe follows the standard expansion law (parameters: $r_* = 0.49$; $\tau_1 = 4 \times 10^4$; $\tau_2 = 4$).

by use of the current derivative

$$\begin{aligned} \nabla_\mu j_\nu &= \frac{i}{\hbar c} \bar{\psi} \cdot [{}^{(w)}\mathscr{H}_\mu \cdot \gamma_\nu - \gamma_\nu \cdot {}^{(w)}\mathscr{H}_\mu] \cdot \psi \\ &= -8\left({}^{(r)}W - \frac{m}{4}\right) S_{\mu\nu} + 3\,{}^{(c)}N b_\mu j_\nu \\ &\quad + 32\,{}^{(r)}W b_\mu b^\lambda S_{\lambda\nu} + {}^{(c)}N\{j_\mu b_\nu - G_{\mu\nu}(b^\lambda j_\lambda)\} \end{aligned} \tag{7.6}$$

and of the Hubble flow

$$\nabla_\mu b_\nu = H \mathscr{B}_{\mu\nu} \tag{7.7}$$

that the charge density is indeed inhomogeneous:

$$\mathscr{B}^\nu{}_\mu\, \partial_\nu I = ({}^{(c)}N + H) \mathscr{B}^\nu{}_\mu j_\nu - 8\left({}^{(r)}W - \frac{m}{4}\right) b^\nu S_{\mu\nu} \tag{7.8}$$

However, the time dependence of I is found as

$$b^\mu\, \partial_\mu I = \dot{I} = 3\,{}^{(c)}N I \tag{7.9}$$

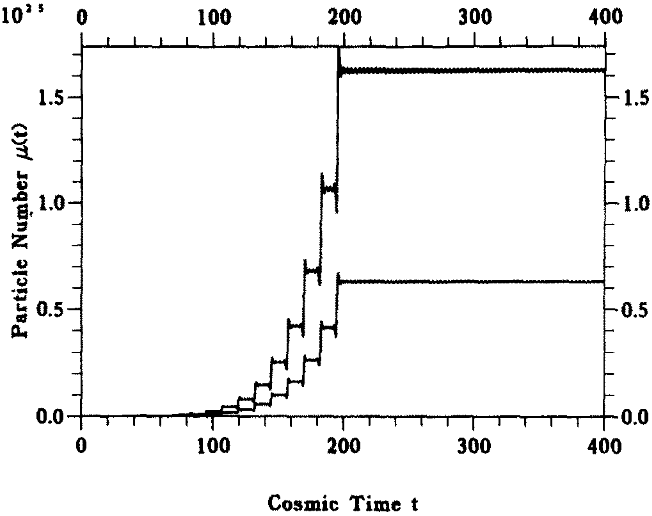

Fig. 2. Unlimited matter production. For the hypothetical expansion law (7.4), solutions exist for the Dirac system (7.1), (7.3) with an arbitrarily large particle number μ (here $\mu \approx 10^{24}$) (lower curve). The matter production occurs only during a bounce (⟿ minimal radius). Through the hypothetical phase transition, the huge matter energy is assumed to be converted into ordinary particles ($\mathscr{P} \geq 0$) and the standard expansion begins (initial conditions for $t \to 0$; $\mu \sim t^3$, $\chi \sim \pi/2 + t/2$). If the polarization effect is included ($\xi \neq 0$), the matter production is even more violent (upper curve, $\mu \approx 10^{25}$).

and thus, though spatially inhomogeneous, the local ratio of charge density I and scalar density ρ is time independent [cf. (7.5)]. As a consequence, we conclude that the intrinsic velocity κ is inhomogeneous here, in contrast to the closed and flat cases, and a closer inspection indeed yields (Sorg, 1993)

$$\kappa = \frac{1}{2} r \tag{7.10}$$

where r is the radial coordinate of the well-known parametrization of the Robertson–Walker line element (Misner *et al.*, 1973)

$$ds^2 = d\theta^2 - \mathscr{R}^2\{dr^2 + \sinh^2 r\,(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)\} \tag{7.11}$$

In a similar way, one shows that the spinor product z is also inhomogeneous

$$z = \cos\vartheta \tag{7.12}$$

and it remains to look for the triad $\{\tilde{\mathscr{E}}_i\}$. Since it is orthogonal to the

Table I

	Closed $\sigma = -1$	Flat $\sigma = 0$	Open $\sigma = +1$
Cosmic form (1.2) for $T_{\mu\nu}$ possible?	No, see (3.9)	Yes, see (6.3)	Yes, see (3.10)
Pressure $\mathscr{P}$	Oscillating	Zero	Oscillating
Energy production	Limited	Zero	Unlimited
Magnetic dipole density	Unidirectional, quasistatic	Unidirectional, quasistatic	Space dependent, quasistatic
Electric dipole density	Oscillating, unidirectional	Oscillating, elliptic polarization	Oscillating, elliptic polarization
Schrödinger trembling suppressible?	No	Yes	No
Current j_μ proportional to Hubble flow b_μ possible?	No	Yes	No
Charge density I homogeneous	Yes	Yes	No
Spinor product z	$0 \leq z \leq 1$	$z = 0$	$z = \cos \vartheta$
Intrinsic velocity κ	Homogeneous, time dependent	Space-time independent	Inhomogeneous, time independent
Angular frequency $\dot{\chi}$	Time dependent	Constant	Time dependent

Hubble flow b_μ, just as the orthonormalized coordinate triad $\{r_\mu, \vartheta_\mu, \varphi_\mu\}$

$$r_\mu = \mathscr{R}(\partial_\mu r) \tag{7.13a}$$

$$\vartheta_\mu = \mathscr{R} \sinh r \, (\partial_\mu \vartheta) \tag{7.13b}$$

$$\varphi_\mu = \mathscr{R} \sinh r \sin \vartheta \, (\partial_\mu \varphi) \tag{7.13c}$$

both must be related to each other by an $\mathcal{O}(3)$ rotation, and indeed one finds

$$\tilde{g}_\mu \equiv -r_\mu \tag{7.14a}$$

$$\tilde{\lambda}_\mu \equiv \vartheta_\mu \tag{7.14b}$$

$$\tilde{\pi}_\mu \equiv \varphi_\mu \tag{7.14c}$$

These kinematical properties of the solution to the Dirac equation in an open RW universe then imply dipole densities of the following form:

$$^{(e)}M_\mu = -\frac{e\hbar}{2Mc} \rho \sinh r \, \{\sin \chi \cdot r_\mu + \sin \vartheta \cdot \cos \chi \cdot \varphi_\mu\} \tag{7.15a}$$

$$^{(m)}M_\mu = -\frac{e\hbar}{2Mc} \rho \{\cos \vartheta \cdot r_\mu - \cosh r \cdot \sin \vartheta \cdot \vartheta_\mu\} \tag{7.15b}$$

Thus, the electric dipole density exhibits here an elliptical polarization in place of the simple linear polarization in a closed universe (5.17a).

8. DISCUSSION

The preceding results clearly demonstrate that any one of the three RW universes $\sigma = 0, \pm 1$ carries a rather distinct physics and it is only the *open* universe which is well suited for the process of cosmic pumping. Furthermore, the latter universe is also capable of "self-pumping," i.e., one does not need to prescribe an oscillating radius $\mathscr{R}(\theta)$ by hand, but one can let these oscillations be practiced by coupling general relativity to the Dirac matter field. By choosing suitable initial conditions, one can indeed obtain solutions of the coupled Dirac–Einstein system with ever-increasing energy density through continuous self-pumping (Mattes *et al.*, 1993). Our results are collected in Table I (here $^{(c)}\tilde{W} \equiv 0$).

REFERENCES

Abbott, L. F., and Pi, S.-Y. (1986). *Inflationary Cosmology*, World Scientific, Singapore.

Blau, S. K., and Guth, A. H. (1987). In *300 Years of Gravitation*, S. W. Hawking and W. Israel, eds., Cambridge University Press, Cambridge.

Guth, A. H. (1981). *Physical Review D*, **23**, 347.

Hawking, S. W. (1990). In *Origins. The Lives and Worlds of Modern Cosmologists*, A. Lightman and R. Brawyer, eds., Harvard University Press, Cambridge, Massachusetts.
Kolb, E. W., and Turner, M. S. (1990). *The Early Universe*, Addison-Wesley, Reading, Massachusetts.
Mattes, M., and Sorg, M. (1993). *Journal of Physics A*, **26**, 3013.
Mattes, M., Ochs, U., and Sorg, M. (1993). Matter production in the early universe, Preprint.
Misner, C. W., Thorne, K. S., and Wheeler, J. A. (1973). *Gravitation*, Freeman, San Francisco.
Ochs, U., and Sorg, M. (1993). Non-singular, cosmological solutions to the coupled Dirac–Einstein equations, Preprint.
Ochs, U., and Sorg, M. (1994). *International Journal of Theoretical Physics*, **9**, 1531.
Penrose, S. (1989). *Difficulties with inflationary cosmology*, In *Fourteenth Texas Symposium on Relativistic Astrophysics*, E. J. Fenyves, ed., New York.
Sorg, M. (1992*a*). Relativistic Schrödinger equations, Preprint.
Sorg, M. (1992*b*). Cosmic spinorfields and the flatness problem, Preprint.
Sorg, M. (1992*c*). Spin precession and expansion of the universe, Preprint.
Sorg, M. (1993). Cosmological principle and primordial spinor fields, *Lett. Math. Phys.*, to appear.